第 1 章

introduction

1.1 Definition of $\theta(z, \tau)$ and its preiodicity in z

The central character in our theory is the analytic function $\theta(z, \tau)$ in 2 variables defined by

$$\theta(z, \tau) = \sum_{n \in \mathbb{Z}} \exp(i\pi n^2 \tau + 2\pi i n z)$$

where $z \in \mathbb{C}, \tau \in \mathcal{H}$.

The series converges absolutely and uniformly on compact sets; in fact, if $|\mathrm{Im} z| < c(\mathrm{Im} z > -c)$ and $\mathrm{Im}\tau > \varepsilon$, then

$$\begin{aligned} |\exp(i\pi n^2 \tau + 2\pi i n z)| &= \exp(-\pi n^2 \mathrm{Im} z) \cdot \exp(-2\pi n \mathrm{Im} z) \\ &\leq \exp(-\pi n^2 \varepsilon) \cdot \exp(2\pi n c) \to 0 \end{aligned}$$

if n_0 is chosen so that

$$\exp(-\pi \varepsilon n_0) \cdot \exp(-2\pi c) < 1$$

then the inequaltiy

$$\exp(-\pi n^2 \varepsilon) \cdot \exp(2\pi n c) \leq \exp(-\pi \varepsilon (n^2 - n n_0))$$

shows that the series converges ant that too very rapidly. (コンパクト空間上 z, t のとり方によらず絶対収束する不等式で抑えた)

We may think of this series as the Fourier series for a function in z, periodic with respect to $z \mapsto z + 1$

$$\theta(z, \tau) = \sum_{n \in \mathbb{Z}} a_n(\tau) \exp(2\pi i n z), a_n(\tau) = \exp(\pi i n^2 \tau)$$

which displays the obvious fact that

$$\theta(z + 1, \tau) = \theta(z, \tau)$$

The precullar form

TBD

1.2 $\theta(x, it)$ as tha fundamental periodic solution to the Heat equation

1.3 The Heisenberg group and theta functions with characteristics

In addition to the standart theta functions discussed so far, there are variants called "theta functions with characteristics" which play a very important role in understanding the functional equation and the identities satisfied by θ, as well as the application of θ to elliptic curves. These are best understood group-theoretically. To explain this, let us fix a, τ and then rephrase the definition of the theta function $\theta(z, t)$ by introducing transformations as follows:

For every holomorphic function $f(z)$ and real numbers a and b, let

$$(S_b f)(z) = f(z+b)$$
$$(T_a f)(z) = \exp(i\pi a^2\tau + 2\pi iaz) f(z + a\tau)$$

Note then that:

$$\begin{aligned}
S_{b_1}(S_{b_2} f) &= S_{b_1+b_2}(f) \\
T_{a_1}(T_{a_2} f) &= \exp(i\pi a_1^2\tau + 2\pi i a_1 z)(T_{a_2} f)(z + a_1\ tau) \\
&= \exp(i\pi a_1^2\tau + 2\pi i a_1 z)\exp(i\pi a_2^2\tau + 2\pi i a_2(z + a_1\tau)) f(z + a_1\tau + a_2\tau) \\
&= \exp(i\pi\tau(a_1^2 + a_2^2 + 2a_1a_2) + 2\pi i(a_1 + a_2)z) f(z + a_1\tau + a_2\tau) = T_{a_1+a_2}(f)
\end{aligned}$$

These are the so called "1-parameter groups", which means continuous homomorphism from $\mathbb{R}$ to groups. Howeve, they do not commute!. We have:

$$\begin{aligned}
S_b(T_a f)(z) &= (T_a)(f(z+b)) \\
&= \exp(\pi a^2 i\tau + 2\pi ia(z+b)) f(z + a\tau + b)
\end{aligned}$$

and

$$\begin{aligned}
T_a(S_b f)(z) &= \exp(\pi a^2 i\tau + 2\pi iaz)(S_b f)(z + a\tau) \\
&= \exp(\pi a^2 i\tau + 2\pi iaz) f(z + a\tau + b)
\end{aligned}$$

and hence

$$S_b \circ T_a = \exp(2\pi iab) T_a \circ S_b \qquad ((*))$$

The group of transformations generated by the T_a's and S_b's is the 3-dimensional group

$$\mathcal{G} = \mathbb{C}_1^* \times \mathbb{R} \times \mathbb{R}, (\mathbb{C}_1^* = \{z \in \mathbb{C} \mid |z| = 1\}$$

where $(\lambda, a, b) \in \mathcal{G}$ stands for the transformation:

$$(U_{(\lambda,a,b)} f(z)) = \lambda(T_a \circ S_b f)(z)$$

This is because,

$$T_{a_1} S_{b_1} T_{a_2} S_{b_2} = \exp(2\pi i a_2 b_1) T_{a_1} T_{a_2} S_{b_1} S_{b_2}$$

$$= \exp(2\pi a_2 b_1) T_{a_1+a_2} S_{b_1+b_2}$$

hence,the group generated by T, S is subset $\mathcal{G}$, and

$$U_{(\lambda,a,b)} = T_{a-1} S_\lambda T_1 S_{b-\lambda}$$

The group law on $\mathcal{G}$ is given by

$$(\lambda, a, b)(\lambda', a', b') = (\lambda\lambda' \exp(2\pi i b a'), a + a', b + b')$$

Note that

$$Z(\mathcal{G}) = \mathbb{C}_1^* = [\mathcal{G}, \mathcal{G}]$$

特に証明がなかったがこれを証明しておくと, $(\lambda, a, b) \in Z(\mathcal{G})$ とすると, 任意の $(\lambda', a', b') \in \mathcal{G}$ に対し,

$$\begin{aligned}(\lambda, a, b)(\lambda', a', b') &= (\lambda\lambda' \exp(2\pi i b a'), a + a', b + b') \\ &= (\lambda', a', b')(\lambda, a, b) \\ &= (\lambda\lambda' \exp(2\pi i b' a), a + a', b + b')\end{aligned}$$

となるので,$\exp(ab' - ba') = 1$ となり,$a = b = 0$ しかありえない. 第 2,3 成分は交換可能なので $[\mathcal{G}, \mathcal{G} \subset \mathbb{C}_1^*$ であり,$a = a' = b = 1, b' = n$ として計算すればすべての $\mathbb{C}_1^*$ が表せることがわかる.

この群は

$$\{1\} \triangleright \mathbb{C}_1^* = [\mathcal{G}, \mathcal{G}] \triangleright G$$

より nilpotent group となる. 上で定めた nilpotent group を **Heisenberg group** という. In fact, the relation * is simly Weyl's integrated form of the Heisenberg commutation relations. Now recall that we have the classical theorem of Von Neumann and Stone which says that $\mathcal{G}$ has a unique ireducible unitary representation in which $(\lambda, 0, 0)$ acts by λ (identity)

ここではそのお気持ちが述べられているが、証明は不明. E を $\mathbb{C}$ 上の正則関数全体とし,$f \in E$ に対し,

$$\|f\| = \int_{\mathbb{C}} \exp(-2\pi y^2/\mathrm{Im}\tau)|f(x+iy)|^2 dxdy$$

と定め,

$$\mathcal{H} = \{f \in E \mid \|f\| < \infty\}$$

とする. 真面目に積分計算すると Unitary は示せる. $\mathcal{H}$ が Hilbert space であることも示せる. 中線定理が成り立つことから内積空間であることがいえ, 点列の極限を計算することで完備性が言える. ただ irreducible かや unique は不明. ただこれはこれ以上扱わない.

To return to θ; note that the subset

$$\Gamma = \{(1, a, b) \in \mathcal{G} \mid a, b \in \mathbb{Z}\}$$

is a subgroup of $\mathcal{G}$, By the characteriuzation of θ in 1.1, we see that, upto scalas, θ is the unique entire function invariant under *Gamma*. Suppose now that ℓ is a positive integer; set $\ell\Gamma = \{(1, \ell a, \ell b)\} \subset \Gamma$ and

$$V_\ell = \{\text{entire functions} f(z) \text{invariant under} \ell\Gamma\}$$

Then, we have the following

Lemma 1.3.1. *An entire function $f(z)$ is in V_ℓ if and only if*

$$f(z) = \sum_{n\in 1/\ell\mathbb{Z}} c_n \exp(\pi i n^2\tau + 2\pi i n z)$$

such that $c_n = c_m$ if $n - m \in \ell\mathbb{Z}$. In particular, $\dim\ell = \ell^2$.

Proof. $f(z) \in V_\ell$ の時, S_ℓ で不変なので,

$$S_\ell(f)(z) = f(\ell + z) = f(z)$$

となるので周期 ℓ を持ち、それについて Fourier Expansion できる.

$$f(z) = \sum_{n\in 1/\ell\mathbb{Z}} c'_n \exp(2\pi i n z)$$

$c'_n = c_n \exp(\pi i n^2\tau)$ としして $f(z)$ に T_ℓ を作用させると,

$$\begin{aligned}T_\ell(f)(z) &= f(z + \ell\tau)\exp(\pi i\ell^2\tau + 2\pi i\ell z)\\ &= \sum c_n \exp(\pi i n^2\tau + 2\pi i n(z + \ell\tau)\exp(\pi i\ell^2\tau + 2\pi i\ell z) \quad = \sum c_n \exp(\pi i(n+\ell)^2\tau + 2\pi i n(z+\ell))\end{aligned}$$

より $c_n = c_{n=\ell}$ となる. 逆は明らか □

For $m \in \mathbb{N}$. let $\mu_m \subset \mathbb{C}_1^*$ be the group of m-th roots of 1. For $\ell \in \mathbb{N}$, let $\mathcal{G}_\ell$ be the finite group defined as

$$\widetilde{\mathcal{G}_\ell} = \mu_{\ell^2} \times 1/\ell\mathbb{Z} \times 1/\ell\mathbb{Z}$$

with group law given by

$$(\lambda, a, b)(\lambda', a', b') = (\lambda\lambda' \exp(2\pi i b a'), a + a', b + b')$$

この時,$\ell\Gamma \subset \widetilde{\mathcal{G}_\ell}$ は正規部分群になるので, それで割ることができ,

$$\mathcal{G}_\ell := \mu_{\ell^2} \times \frac{1}{\ell}\mathbb{Z}/\ell\mathbb{Z} \times \frac{1}{\ell}\mathbb{Z}/\ell\mathbb{Z}$$

となる. Noe the elements $S_{1/\ell}, T_{1/\ell} \in \mathcal{G}$ commute with $\ell\Gamma$(in view of *) and hence act on V_ℓ. This goes down to an action of $\mathcal{G}_\ell$ on V_ℓ; in fact, exactly like $\mathcal{G}$, the generators $S_{1/\ell}$ of $\mathcal{G}_\ell$ act on V_ℓ as follows:

$$S_{1/\ell}(\sum c_n \exp(\pi i n^2\tau + 2\pi i n z)) = \sum c_n \exp(2\pi i n/\ell)\exp(\pi i n^2\tau + 2\pi i n z)$$

and

$$T_{1/\ell} f(z) = f(z + 1/\ell\tau)\exp(\pi i 1/\ell^2\tau + 2\pi)$$

Lemma 1.3.2. *$\mathcal{G}_\ell$ acts irreducibily on V_ℓ.*

TBD

1.4 Projetive Embedding of $C/\mathbb{Z}+\mathbb{Z}\tau$ by means of theta functions

The theta functions $\theta_{a,b}$ defined above have avery important geometric application. Take any $\ell \geq 2$. Let E_τ be the complex torus $\mathbb{C}/\Lambda_\tau$ where $\Lambda_\tau = \mathbb{Z}+\mathbb{Z}\tau$. Let (a_i, b_i) be a set of coset representaiton of $(\frac{1}{\ell}\mathbb{Z}/\mathbb{Z})^2$, $0 \leq i \leq \ell^2-1$. Write $\theta_i = \theta_{a_i,b_i}$. For all $z \in \mathbb{C}$, consider the ℓ^2-tuple

$$(\theta_0(\ell z,\tau),\ldots,\theta_{\ell^2-1}(\ell z,\tau))$$

これは $\mathbb{P}_{\mathbb{C}}^{\ell^2-1}$ の元を定める. さらに実際は $\phi_\ell : E_\tau \to \mathbb{P}^{\ell^2-1}$ を定める.

この Well-defined 性は

- 上の写像が定義域の代表元のとり方によらない. つまり $a \in \Lambda_\tau$ を足した時に定数倍 $(\theta_0(\ell z,\tau),\ldots,\theta_{\ell^2-1}(\ell z,\tau)) = \lambda(\theta_0(\ell z+a,\tau),\ldots,\theta_{\ell^2-1}(\ell z+a,\tau))$ になっている。(実際は生成元の場合だけ示せばよい)
- 値域が射影空間の外に出ない. つまり, 全て同時に 0 とならない.

まず最初の代表元のとり方によらないことを見ておく. $\theta_{a,b}(z+\ell,\tau) = \sum_{n\in\mathbb{Z}} \exp(\pi i(a+n)^2\tau + 2\pi i(n+a)(z+\ell+b))$ であり,$a\ell \in \mathbb{Z}$ より, $\theta_{a,b}(z,\tau)$ と一致する.

$$\begin{aligned}
\theta_{a,b}(z+\ell\tau,\tau) &= \sum_{n\in\mathbb{Z}} \exp(\pi i(a+n)^2\tau + 2\pi i(n+a)(z+\ell\tau+b)) && \text{by definition}\\
&= \sum_{n\in\mathbb{Z}} \exp(\pi i(a^2+n^2+2an+2n\ell+2a\ell)\tau + 2\pi i(n+a)(z\tau+b))\\
&= \sum_{n\in\mathbb{Z}} \exp(\pi i((a+n+\ell)^2-\ell^2)\tau + 2\pi i(n+a)(z\tau+b))\\
&= \sum_{n\in\mathbb{Z}} \exp(\pi i((a+n+\ell)^2-\ell^2)\tau + 2\pi i\,((n+a+\ell)(z\tau+b)-\ell(z\tau+b))\\
&= \sum_{n\in\mathbb{Z}} \exp(\pi i((a+n+\ell)^2-\ell^2)\tau + 2\pi i\,((n+a+\ell)(z\tau+b)-\ell z\tau)\\
&= \sum_{n\in\mathbb{Z}} \exp(-\pi i\ell^2 - 2\pi i\ell z\tau)\exp(\pi i((a+n+\ell)^2\tau + 2\pi i(n+a+\ell)(z\tau+b))\\
&= \lambda\sum_{n\in\mathbb{Z}} \exp(-\pi i\ell^2 - 2\pi i\ell z\tau)\exp(\pi i((a+n+\ell)^2\tau + 2\pi i(n+a+\ell)(z\tau+b))\\
&= \lambda\sum_{n\in\mathbb{Z}} \exp(-\pi i\ell^2 - 2\pi i\ell z\tau)\exp(\pi i((a+n)^2\tau + 2\pi i(n+a)(z\tau+b)) && n=n+\ell\text{の置き換え}
\end{aligned}$$

ただし,$\exp(-\pi i\ell^2 - 2\pi i\ell z\tau = \lambda$ とおいた. これより a, b のとり方によらず一定変化するため,well-defined である.

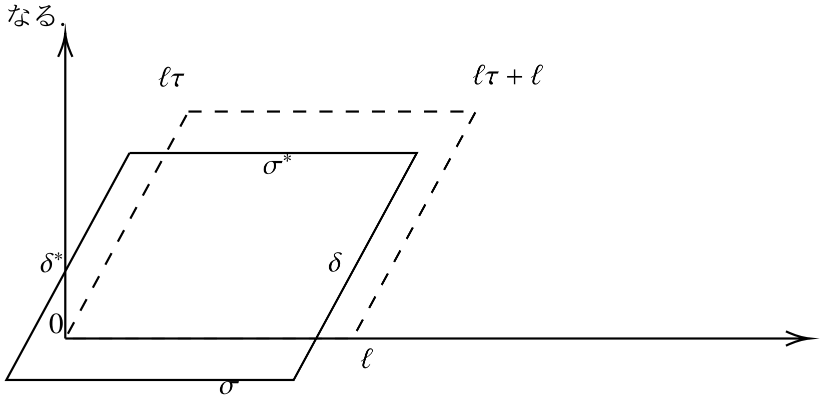
全て 0 にならないことは以下からわかる.

Lemma 1.4.1. *0 でない* $f \in V_\ell$ は $\mathbb{C}/\ell\Lambda_\tau$ の基本領域上に ℓ^2 個のゼロ点を持つ. $\theta_{a,b}$ の場合, それは $(a+p+\frac{1}{2}, b+q+\frac{1}{2})(p,q\in\mathbb{Z})$ となる. ここから特に $i \neq j$ の時,θ_i,θ_j のゼロ点は異なる.

Proof. f の零点は軸上にないように必要ならば平行移動して考える.(0 でない正則関数が有界区間上で 0 となる個数は有限個なので) この時, ゼロ点の個数は

$$\text{number of zeros of f} = \frac{1}{2\pi i}\int_{\sigma+\sigma^*+\delta+\delta^*}\frac{f'}{f}dz$$

となる.



$f(z+\ell) = f(z)$ より $\int_{\sigma+\sigma^*} f = 0$ となる. $f(z+\ell\tau) = const\exp(-2\pi i\ell z)f(z)$ となるので,$f(z+\ell\tau)' = const(-2\pi i\ell\exp(-2\pi i\ell z)f(z) + \exp(-2\pi i\ell z)f'(z))$ となり.

$$\begin{aligned}
\int_{\delta+\delta^*}\frac{f'}{f} &= \int_{\delta}\frac{f'(z)}{f(z)}dz + \int_{-\delta}\frac{f'(z+\ell\tau)}{f(z+\ell\tau)}dz \\
&= \int_{\delta}\frac{f'(z)}{f(z)} - \frac{const(-2\pi i\ell\exp(-2\pi i\ell z)f(z)+\exp(-2\pi i\ell z)f'(z))}{const\exp(-2\pi i\ell z)f(z)} \\
&= \int_{\delta}\frac{f'(z)}{f(z)} - \frac{(-2\pi i\ell f(z)+f'(z))}{f(z)} \\
&= \int_{\delta}-\frac{-2\pi i\ell f(z)}{f(z)} \\
&= 2\pi i\ell^2
\end{aligned}$$

よって ℓ^2 個であることが言えた. $\theta(z,\tau)$ は $\mathbb{C}/\Lambda_\tau$ 上一つだけゼロ点を持つ.($\ell = 1$).

$$\begin{aligned}
\theta_{1/2,1/2}(-z,\tau) &= \sum_{n\in\mathbb{Z}}\exp\left(\pi i(n+1/2)^2\tau + 2\pi i(n+1/2)(-z+1/2)\right) \\
&= \sum_{m\in\mathbb{Z}}\exp\left(\pi i(-m-1/2)^2\tau + 2\pi i(-m-1/2)(-z+1/2)\right) \qquad m=-n-1 \\
&= \sum_{m\in\mathbb{Z}}\exp\left(\pi i(m+1/2)^2\tau + 2\pi i(m+1/2)(z+1/2) - 2\pi i(m+1/2)\right) \\
&= -\sum_{m\in\mathbb{Z}}\exp\left(\pi i(m+1/2)^2\tau + 2\pi i(m+1/2)(z+1/2)\right) \\
&= -\theta_{1/2,1/2}(z,\tau)
\end{aligned}$$

これから $\theta_{1/2,1/2}$ は $z=0$ でゼロになる.(これは具体的に計算して 0 を示すのは難しいんだろうか) 後は周期性から計算すればわかる. □

この後 ϕ_ℓ の群作用や Embedding になっていることを見る.(TBD)

1.5　Riemann's theta relations

記号を改めてここで定義する.

$\theta(z,\tau) := \sum_{n\in\mathbb{Z}} \exp(i\pi n^2\tau + 2i\pi z)$, $\Lambda_\tau := \mathbb{Z}+\mathbb{Z}\tau$, $S_a\theta(z,\tau) = \theta(z+a,\tau)$, $T_a\theta(z,\tau) = \exp(\pi i a^2\tau + 2\pi i a z)\theta(z+a\tau,\tau)$

A を $A^t A = m^2 I_n$ となる行列とする. 例えば

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

とすると $A^t A = 4I_4$ となる. これは

$$(x+y+u+v)^2 + (x+y-u-v) + (x-y+u-v)^2 + (x-y-u+v)^2 = 4(x^2+y^2+u^2+v^2)$$

を定める. 式が複雑になるため $\theta(z) := \theta(z,\tau)$, $\Lambda := \Lambda_\tau$ と定める.

これらを使っていくつか式を算出する.

$$B(0) := \theta(x)\theta(y)\theta(u)\theta(v) = \sum_{n,m,p,q\in\mathbb{Z}} \exp\left(\pi i(\sum n^2)\tau + 2\pi i(\sum xn)\right)$$

ただし,$\sum xn = xn + ym + up + vq$ であり,x, n を走る和は同様に表記する.

$$\begin{aligned} B\left(\frac{1}{2}\right) :=& \theta(x+\frac{1}{2})\theta(y+\frac{1}{2})\theta(u+\frac{1}{2})\theta(v+\frac{1}{2}) \\ =& \sum_{n,m,p,q\in\mathbb{Z}} \exp\left(\pi i(\sum n^2)\tau + 2\pi i(\sum xn) + \pi i \sum n\right) \qquad \text{ただ x+1/2 を展開しただけ} \end{aligned}$$

$$\exp(\pi i n^2\tau + 2\pi i n(x+\frac{1}{2}\tau)) = \exp(\pi i\tau(n+\frac{1}{2})^2 - \frac{1}{4}\pi i\tau + 2\pi i(n+\frac{1}{2})x - \pi i x \qquad \text{平方完成}$$

$$\exp(1/4\pi i\tau + \pi i x + \pi i n^2\tau + 2\pi i n(x+\frac{1}{2}\tau)) = \exp(\pi i\tau(n+\frac{1}{2})^2 + 2\pi i(n+\frac{1}{2})x \qquad \text{負の項を移項}$$

なので, これの和を取ると,

$$\begin{aligned} B\left(\frac{1}{2}\tau\right) :=& \exp\left(\pi i(\tau + \sum x)\right)\theta(x+\frac{1}{2})\theta(y+\frac{1}{2}\tau)\theta(u+\frac{1}{2}\tau)\theta(v+\frac{1}{2}\tau) \\ =& \sum_{n,m,p,q\in\mathbb{Z}} \exp\left(\pi i\left(\sum (n+1/2)^2\right)\tau + 2\pi i\left(\sum x(n+1/2)\right)\right) \end{aligned}$$

$$\exp(\pi i n^2\tau + 2\pi i n(x+\frac{1}{2}+\frac{1}{2}\tau)) = \exp(\pi i\tau(n+\frac{1}{2})^2 - \frac{1}{4}\pi i\tau + 2\pi i(n+\frac{1}{2})x - \pi i x + \pi i n \qquad \text{平方完成}$$

$$\exp(1/4\pi i\tau + \pi i x + \pi i n^2\tau + 2\pi i n(x+\frac{1}{2}+\frac{1}{2}\tau)) = \exp(\pi i\tau(n+\frac{1}{2})^2 + 2\pi i(n+\frac{1}{2})x + \pi i n) \qquad \text{負の項を移項}$$

$$\begin{aligned} B\left(\frac{1}{2}+\frac{1}{2}\tau\right) :=& \exp\left(\pi i(\tau + \sum x)\right)\theta(x+\frac{1}{2}+\frac{1}{2}\tau)\theta(y+\frac{1}{2}+\frac{1}{2}\tau)\theta(u+\frac{1}{2}+\frac{1}{2}\tau)\theta(v+\frac{1}{2}+\frac{1}{2}\tau) \\ =& \sum_{n,m,p,q\in\mathbb{Z}} \exp\left(\pi i\left(\sum (n+1/2)^2\right)\tau + 2\pi i\left(\sum x(n+1/2)\right) + \pi i \sum n\right) \end{aligned}$$

$B(1/2) = \exp(\pi i \sum n)B(0), B(1/2 + 1/2\tau) = \exp(\pi i \sum n)B(1/2)$ になることと

$$B(0) + B(1/2\tau) = \sum_{n,m,p,q\in 1/2\mathbb{Z}} \exp\left(\pi i\left(\sum n^2\right)\tau + 2\pi i\left(\sum xn\right)\right)$$

となり, $\exp(\pi i \sum n)$ は $\sum$ が偶数のときは 2 倍になり, 奇数の場合は消えるので、

$$\sum_{\eta=0,1/2,1/2\tau 1/2+1/2\tau} B(\eta) = 2\sum_{n,m,p,q\in 1/2\mathbb{Z}} \exp\left(\pi i\left(\sum n^2\right)\tau + 2\pi i\left(\sum xn\right)\right)$$

ただし,n, m, p, q は全て整数か全て $1/2 + \mathbb{Z}$ の元でありさらに合計が偶数になるところを走る.

$$\begin{aligned} n_1 &= \frac{1}{2}(n+m+p+q) & x_1 &= \frac{1}{2}(x+y+u+v) \\ m_1 &= \frac{1}{2}(n+m-p-q) & y_1 &= \frac{1}{2}(x+y-u-v) \\ p_1 &= \frac{1}{2}(n-m+p-q) & u_1 &= \frac{1}{2}(x-y-u+v) \\ q_1 &= \frac{1}{2}(n-m-p+q) & v_1 &= \frac{1}{2}(x-y-u+v) \end{aligned}$$

とすると, $\sum n^2 = \sum n_1^2, \sum xn = \sum x_1 n_1$ となるので,

$$\begin{aligned} \sum_{\eta=0,1/2,1/2\tau 1/2+1/2\tau} B(\eta) &= 2\sum_{n,m,p,q\in 1/2\mathbb{Z}} \exp\left(\pi i\left(\sum n^2\right)\tau + 2\pi i\left(\sum xn\right)\right) \\ &= 2\sum_{n_1,m_1,p_1,q_1} \exp(\pi i \sum n_1^2)\tau + 2\pi i(\sum x_1 n_1)) \end{aligned}$$

これより, 以下の関係式が得られる.

$$(R_1): \sum_{\eta=0,1/2,1/2\tau 1/2+1/2\tau} e_\eta \theta(x+\eta)\theta(y+\eta)\theta(z+\eta)\theta(v+\eta) = 2\sum_{n,m,p,q\in 1/2\mathbb{Z}} \exp\left(\pi i\left(\sum n^2\right)\tau + 2\pi i\left(\sum xn\right)\right)$$

これを $\theta_{a,b}$ を用いて表す. $\theta_{a,b} = T_a S_b \theta = \exp(\pi i a^2\tau + 2\pi i a(z+b))\theta(z + a\tau + b, \tau)$ なので,

$$\begin{aligned} \theta_{0,0} &= \theta(z,\tau) \\ \theta_{0,\frac{1}{2}} &= \theta(z+\frac{1}{2},\tau) \\ \theta_{\frac{1}{2},0} &= \exp(\pi i\frac{1}{4} + \pi i z)\theta(z+1/2\tau,\tau)) \\ \theta_{\frac{1}{2},\frac{1}{2}} &= \exp(\pi i\tau/4 + \pi i(z+\frac{1}{2}))\theta(z+\frac{1}{2}(1+\tau),\tau) \end{aligned}$$

である. これらを $\theta_{0,0}, \theta_{0,1}, \theta_{1,0}, \theta_{1,1}$ と表す.

これらには以下の関係がある.

$$\theta_{1,1}(-z,\tau) = \sum_{n\in\mathbb{Z}} \exp\left(\pi i(n+1/2)^2\tau + 2\pi i(n+1/2)(-z+1/2)\right)$$

$$
\begin{aligned}
&= \sum_{m\in\mathbb{Z}} \exp\left(\pi i(-m-1/2)^2\tau + 2\pi i(-m-1/2)(-z+1/2)\right) \qquad m=-n-1\\
&= \sum_{m\in\mathbb{Z}} \exp\left(\pi i(m+1/2)^2\tau + 2\pi i(m+1/2)(z+1/2) - 2\pi i(m+1/2)\right)\\
&= -\sum_{m\in\mathbb{Z}} \exp\left(\pi i(m+1/2)^2\tau + 2\pi i(m+1/2)(z+1/2)\right)\\
&= -\theta_{1,1}(z,\tau)
\end{aligned}
$$

また, ほかは以下となる.

$$
\begin{aligned}
\theta_{0,0}(-z,\tau) &= \sum \exp(\pi i(-n)^2\tau + 2\pi i(-n)z)) = \theta_{0,0}(z,\tau)\\
\theta_{0,1}(-z,\tau) &= \sum \exp(\pi i(-n)^2\tau + 2\pi in(-z+\frac{1}{2})) = \sum \exp(\pi i(-n)^2\tau + 2\pi i(-n)(z+1/2)) + 2n\pi i) = \theta_{0,1}(z,\tau)\\
\theta_{1,0}(-z,\tau) &= \sum \exp(\pi i(n+\frac{1}{2})^2\tau - 2\pi i(n+\frac{1}{2})z) = \theta_{1,0}(z,\tau)\text{n=-n-1}
\end{aligned}
$$

これを使うと以下の関係式が得られる.

$$
(R_2) : \sum \theta_{i,j}(x)\theta_{i,j}(y)\theta_{i,j}(u)\theta_{i,j}(v) = 2\theta_{0,0}(x_1)\theta_{0,0}(y_1)\theta_{0,0}(u_1)\theta_{0,0}(v_1)
$$

x を $x+1$ に置き換えると, $\sum \exp(\pi in^2\tau + 2\pi inz + 2\pi in) = \sum \exp(\pi in^2\tau + 2\pi inz)$, $\exp(\pi i\frac{1}{4} + \pi i(z+1)) = -\exp(\pi i\frac{1}{4} + \pi iz)$ となるので,

$$
\begin{aligned}
(R_3) : &\theta_{0,0}(x)\theta_{0,0}(y)\theta_{0,0}(u)\theta_{0,0}(v) + \theta_{0,1}(x)\theta_{0,1}(y)\theta_{0,1}(u)\theta_{0,1}(v)\\
&- \theta_{1,0}(x)\theta_{1,0}(y)\theta_{1,0}(u)\theta_{1,0}(v) - \theta_{1,1}(x)\theta_{1,1}(y)\theta_{1,1}(u)\theta_{1,1}(v)\\
&= 2\theta_{0,1}(x_1)\theta_{0,1}(y_1)\theta_{0,1}(u_1)\theta_{0,1}(v_1)
\end{aligned}
$$

となる. 同様に $x = x+\tau$ とすると,

$$
\begin{aligned}
\exp(\pi i\tau + 2\pi ix)\theta_{0,0}(z+\tau,\tau) &= \sum \exp(\pi i\tau + 2\pi ix)\exp(\pi i(n)^2\tau + 2\pi i(n)(x+\tau)) = \theta_{0,0}(x,\tau)\\
&= \sum \exp(\pi i(n+1)^2 + 2\pi(n+1)x) = \theta_{0,0}(x,\tau)\\
\exp(\pi i\tau + 2\pi ix)\theta_{0,1}(x+\tau,\tau) &= \sum \exp(\pi i(n+1)^2\tau + 2\pi in(x+\frac{1}{2}+\tau)\\
&= \sum \exp(\pi i(n+1)^2\tau + 2\pi i(n+1)(x+1/2)) - \pi i) = -\theta_{0,1}(z,\tau)\\
\exp(\pi i\tau + 2\pi ix)\theta_{1,0}(x+\tau,\tau) &= \sum \exp(\pi i\tau + 2\pi ix)\exp(\pi i\tau/4 + \pi i(x+\tau))\exp(\pi in^2\tau + 2\pi in(x+\frac{3}{2}\tau)\\
&= \sum \exp(\pi i\tau + 2\pi ix + \pi i\tau/4 + \pi i(x+\tau)\\
&+ \pi i\tau(n+1)^2 - \pi i\tau + 2\pi i(n+1)(x+\frac{1}{2}\tau) - 2\pi i(x+\frac{1}{2}\tau))\\
&= \sum \exp(\pi i\tau/4 + \pi ix)\exp(\pi\tau(n+1)^2 + 2\pi i(n+1)(x+\frac{1}{2}\tau) = \theta_{1,0}(x)\\
\exp(\pi i\tau + 2\pi ix)\theta_{1,1}(x+\tau,\tau) &= -\theta_{1,1}(x)
\end{aligned}
$$

となる.

$2x = x_1 + y_1 + u_1 + v_1$ なので, $\exp(\pi i\tau + 2\pi ix) = \exp\left(\sum(\pi i\tau/4 + \pi ix_1)\right)$ となる. よって

$$
\begin{aligned}
(R_4) : &\theta_{0,0}(x)\theta_{0,0}(y)\theta_{0,0}(u)\theta_{0,0}(v) - \theta_{0,1}(x)\theta_{0,1}(y)\theta_{0,1}(u)\theta_{0,1}(v)\\
&+ \theta_{1,0}(x)\theta_{1,0}(y)\theta_{1,0}(u)\theta_{1,0}(v) - \theta_{1,1}(x)\theta_{1,1}(y)\theta_{1,1}(u)\theta_{1,1}(v)
\end{aligned}
$$

$$=2\theta_{0,1}(x_1)\theta_{0,1}(y_1)\theta_{0,1}(u_1)\theta_{0,1}(v_1)$$

となる.

同様に x を $x+\tau+1$ に置き換えることで,

$$\begin{aligned}(R_5):\ &\theta_{0,0}(x)\theta_{0,0}(y)\theta_{0,0}(u)\theta_{0,0}(v)-\theta_{0,1}(x)\theta_{0,1}(y)\theta_{0,1}(u)\theta_{0,1}(v)\\ &-\theta_{1,0}(x)\theta_{1,0}(y)\theta_{1,0}(u)\theta_{1,0}(v)+\theta_{1,1}(x)\theta_{1,1}(y)\theta_{1,1}(u)\theta_{1,1}(v)\\ =&2\theta_{1,1}(x_1)\theta_{1,1}(y_1)\theta_{1,1}(u_1)\theta_{1,1}(v_1)\end{aligned}$$

が得られる.

また x を $x+1/2, x+1/2\tau$ とすることで同様に様々な公式が得られる.

またこうして得られた式から $x=y, u=v$ とすると,$x_1=x+v, y_1=x-v, u_1=0, v_1=0$ となる. $\theta_{1,1}(0)=0$ より.R_5 の右辺は 0 になる. これから等式を変形すると

$$\theta_{0,0}(x)^2\theta_{0,0}(u)^2+\theta_{1,1}(x)^2\theta_{1,1}(u)^2=\theta_{0,1}(x)^2\theta_{0,1}(u)^2+\theta_{0,1}(x)^2\theta_{0,1}(u)^2$$

となり, また $R2+R5$ より,

$$\theta_{0,0}(x)^2\theta_{0,0}(u)^2+\theta_{1,1}(x)^2\theta_{1,1}(u)^2=\theta_{0,1}(x)^2\theta_{0,1}(u)^2+\theta_{1,0}(x)^2\theta_{1,0}(u)^2=\theta_{00}(x+u)\theta_{00}(x-u)\theta_{00}(0)^2$$

となる. 同様に x, u に対して $x+u, x-u$ に関する 関係式が得られる.

こうした関係式に $u=0$ を代入することで,

$$\theta_{0,1}(x)^2\theta_{0,1}(0)^2+\theta_{1,0}(x)^2\theta_{1,0}(0)^2=\theta_{00}(x)^2\theta_{00}(0)^2$$

が得られる. 同様に (いろいろ計算すると)

$$\theta_{0,1}(x)^2\theta_{1,0}(0)^2-\theta_{1,0}(x)^2\theta_{0,1}(0)^2=\theta_{11}(x)^2\theta_{00}(0)^2$$

が得られる.

上の関係式に $x=0$ を代入すると

$$\theta_{0,1}(0)^4+\theta_{1,0}(0)^4=\theta_{00}(0)^4$$

が得られ, これは **ヤコビの恒等式**と呼ばれる.

1.6 Doubly periodic meromorphic functions via $\theta(z,\tau)$

この章では 4 つの手段で E_τ 上の meromorphic function を作る. これは $\mathbb{C}$ 上の有理型関数であって,Λ_τ 上周期的であればよい.

1.6.1 By restriction of rational functions from $\mathcal{P}^3$

$\phi_\ell:\mathbb{C}/\Lambda_\tau\to\mathbb{P}^{\ell^2-1}, (\theta_0(\ell z,\tau),\dots,\theta_{\ell^2-1}(\ell z,\tau)$ は embedding だったので, $\ell=2$ の場合にも embedding になっている。 そこで,$\mathbb{C}\to\mathbb{P}^1, z\mapsto\frac{\theta_{a,b}}{\theta_{0,0}}(a,b\in\{0,1\})$ は meromorphic になる.

1.6.2 As quotients of products of translates of $\theta(z)$ itself

$a_1,\dots,a_k,b_1,\dots,b_k$ を $\sum a_i = \sum b_i$ とする. この時

$$\prod_{1\le i\le k}\frac{\theta(z-a_i)}{\theta(z-b_i)}$$

は Λ_τ 上周期的である. それは,

$$\begin{aligned}\theta(z+1) &= \sum \exp(\pi i n^2\tau + 2\pi i n(z+1))\\ &= \sum \exp(\pi i n^2\tau + 2\pi i n(z))\\ &= \theta(z)\\ \theta(z+\tau) &= \sum \exp(\pi i n^2\tau + 2\pi i n(z+\tau))\\ &= \sum \exp(\pi i(n+1)^2\tau - \pi i\tau + 2(n+1)z\pi i - 2z\pi i)\\ &= \sum \exp(-\pi i\tau - 2\pi i z)\theta(z)\end{aligned}$$

より,$\theta(z-a+1)=\theta(z-a), \theta(z-a+\tau)=\exp(-\pi i\tau - 2\pi i(z-a))\theta(z-a)$ となる. よって,

$$\prod_{1\le i\le k}\frac{\theta(z-a_i+1)}{\theta(z-b_i+1)} = \prod_{1\le i\le k}\frac{\theta(z-a_i)}{\theta(z-b_i)}$$

かつ

$$\begin{aligned}\prod_{1\le i\le k}\frac{\theta(z-a_i+\tau)}{\theta(z-b_i+\tau)} &= \prod_{1\le i\le k}\frac{\exp(-\pi i\tau - 2\pi i(z-a_i))\theta(z-a_i)}{\exp(-\pi i\tau - 2\pi i(z-b_i))\theta(z-b_i)}\\ &= \prod_{1\le i\le k}\frac{\exp(2\pi i a_i)\theta(z-a_i)}{\exp(2\pi i b_i)\theta(z-b_i)}\\ &= \exp(2\pi i\sum(a_i-b_i))\prod_{1\le i\le k}\frac{\theta(z-a_i)}{\theta(z-b_i)}\\ &= \prod_{1\le i\le k}\frac{\theta(z-a_i)}{\theta(z-b_i)}\end{aligned}$$

となる. よって Λ_τ 上周期的な有理型関数になる.

1.6.3 Second logarithmic derivatives

$\log\theta(z+1) = \log\theta(z), \log\theta(z+\tau) = \log\theta(z) - (\pi i\tau - 2\pi i z)$ となるので,

$$\frac{d^2}{dz^2}\log\theta(z+\tau) = \frac{d^2}{dz^2}\log\theta(z)$$

とり, 二重周期関数である. また

$$\wp(z) = -\frac{d^2}{dz^2}\log\theta_{1,1}(z) + const$$

となることを示す. $\wp$ 関数の定義を思い出すと以下の形であった.

$$\wp(z) = \sum_{(n,m)\neq 0}\frac{1}{(z-n\omega_1-m\omega_2)^2} - \frac{1}{(n\omega_1+m\omega_2)^2} + \frac{1}{z^2}$$

これは二重周期を持ち $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ 上に二次の極を持つ.

f が偶関数であって,$\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ に一位のゼロ点を持ち,$\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ 以外で零点を持たない二重周期関数の場合, $f'(\omega_1), f'(\omega_2) \neq 0$ なので,$\frac{d}{dz}\log f = \frac{f'}{f}$ は ω_1, ω_2 で一次の極を持つ. また

$$\frac{d^2}{dz^2}\log f = \frac{ff'' - f'^2}{f^2}$$

であり, $\omega \in \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ に対し, 二次の極を持つ. 二次の極のローラン級数展開したときの z^{-2} 次の係数を求めたい. f は原点の近傍で正則なので, テイラー展開し $f(z) = \sum a_n z^n$ とする.$a_0 = 0, a_1 \neq 0$ となり,

$$\frac{... - (\sum_{n\geq 1} na_n z^{n-1})^2}{z^2(\sum_{n\geq 1} a_n z^{n-1})^2}$$

... は定数項がないので,z^{-2} の係数は-1 となる. 他 $\omega \in \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ についても同様. f/f' も meromorphic なので,$\frac{d^2}{dz^2}\log f$ の-1 乗の係数は 0 になる. よって $\frac{d^2}{dz^2}\log f + \mathfrak{p}$ は全域で極を持たない二重周期関数なので, 定数関数となることがわかる. $\theta_{1,1}$ の定理 4・1 から $\mathbb{Z} + \mathbb{Z}\tau$ 上一位のゼロ点のみを持つので f の条件を満たし, 上の関係式を得る.

1.6.4 Sums of first logarithmic derivatives

$a_i \in \mathbb{C}, \lambda_i \in \mathbb{C}$ で $\sum \lambda_i = 0$ とする.

この時

$$\frac{d}{dz}\log\theta(z+\tau) = \frac{d}{dz}\log\theta(z) - 2\pi i$$

となるので,

$$\begin{aligned}\sum \lambda_i \frac{d}{dz}\log\theta(z - a_i + \tau) &= \sum \lambda_i(\frac{d}{dz}\log\theta(z - a_i) - 2\pi i) \\ &= \sum \lambda_i \frac{d}{dz}\log\theta(z - a_i)\end{aligned}$$

となるので, 周期的な関数である.

1 番目と 2 番目を関係を見る.$\theta_{ab}(2z)$ を θ の積で表す. 例えば前回やった関係式 R18($\theta^x_{ij} = \theta_{i,j}(x)$ と同じにより,

$$\theta^x_{00}\theta^y_{01}\theta^u_{10}\theta^v_{11} + \theta^x_{01}\theta^y_{00}\theta^u_{11}\theta^v_{10} + \theta^x_{10}\theta^y_{11}\theta^u_{00}\theta^v_{01} + \theta^x_{11}\theta^y_{10}\theta^u_{01}\theta^v_{00} = 2\theta^{x_1}_{11}\theta^{y_1}_{10}\theta^{u_1}_{01}\theta^{v_1}_{00}$$

ただし,x_1 等は以下で定めている.

$$x_1 = \frac{1}{2}(x+y+u+v), y_1 = \frac{1}{2}(x+y-u-v), u_1 = \frac{1}{2}(x-y+u-v), v_1 = \frac{1}{2}(x-y-u+v)$$

に $x = y = u = v = z$ とすると, $x_1 = 2z, y_1 = 0, u_1 = 0, v_1 = 0$ より

$$2\theta_{11}(2z)\theta_{10}(0)\theta_{01}(0)\theta_{00}(0) = 4\theta_{00}(z)\theta_{01}(z)\theta_{10}(z)\theta_{11}(z)$$

となる. すいません.$\theta_{00}(2z)$ を積で表すのが難しくて...

2 番目と 3 番目の関係を見る. また A10

$$\theta_{11}(x+u)\theta_{11}(x-u)\theta_{00}^2(0) = \theta_{11}^2(x)\theta_{00}^2(u) - \theta_{00}^2(x)\theta_{11}^2(u)$$

に対し u で 2 回微分すると,

$$\begin{aligned}(\theta_{11}(x+u)\theta_{11}(x-u)\theta_{00}^2(0))'' &= ((\theta_{11}'(x+u)\theta_{11}(x-u)\theta_{00}^2(0) - (\theta_{11}(x+u)\theta_{11}'(x-u)\theta_{00}^2(0))' \\ &= \theta_{11}''(x+u)\theta_{11}(x-u)\theta_{00}^2(0) - 2(\theta_{11}'(x+u)\theta_{11}'(x-u)\theta_{00}^2(0)) + \theta_{11}(x+u)\theta_{11}''(x-u)\theta_{00}^2(0) \\ &= 2\theta_{11}^2(x)(\theta_{00}'^2(u) + \theta_{00}(u)\theta_{00}''(u)) - 2\theta_{00}^2(x)(\theta_{11}'^2(u) + \theta_{11}(u)\theta_{11}''(u))\end{aligned}$$

これに $x = z, u = 0$ を代入する.

$$2\theta_{11}''(z)\theta_{11}(z)\theta_{00}^2(0) - 2\theta_{11}'(z)^2\theta_{00}^2(0) = 2\theta_{11}^2(z)(\theta_{00}(0)\theta_{00}''(0)) - 2\theta_{00}(z)^2\theta_{11}'(0)^2$$

となる. ただし,$\theta_{00}'(0) = 0, \theta_{11}''(0) = 0$ を使った.

$$\begin{aligned}\frac{d^2}{dz^2}\log\theta_{11} &= \frac{\theta_{11}\theta_{11}'' - \theta_{11}'^2}{\theta_{11}^2} \\ = \frac{\theta_{11}^2(z)(\theta_{00}(0)\theta_{00}''(0)) - \theta_{00}(z)^2\theta_{11}'(0)^2}{\theta_{00}^2(0)\theta_{11}^2} \\ &= \frac{\theta_{00}''(0)}{\theta_{00}(0)} - \frac{\theta_{00}(z)^2\theta_{11}'(0)^2}{\theta_{00}^2(0)\theta_{11}^2}\end{aligned}$$

$\wp$ の微分方程式を最後に導いているが, これはローラン級数展開して係数を調整して, 正則な二重周期なので、定数という関係を使う. つまり原点の近傍で $\wp(z) = \frac{1}{z^2} + az^2 + bz^4 + ...$ と表わせ,

$$\wp'(z) = -\frac{2}{z^3} + 2az + 4bz^3...$$

より,$\wp'(z)^2 - 4\wp(z)^3 + 20a\wp(z) = const$ となる.

1.7 The functional equation of $\theta(z,\tau)$

So far we have concentrated on the behaviour of $\theta(z,\tau)$ as a function of z. Its behaviour as a function of τ is also extremely beautiful, but rather deeper and more subtle.

To be precise, fix any

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

and assume that ab, cd are even and $c \geq 0$. Consider the function $\theta((c\tau+d)y,\tau)$. Clearly, when y is replaced by $y+1$, the function is unchanged except for an exponential factor.

今

$$\Psi(y,\tau) = \exp(\pi i c(c\tau+d)y^2)\theta((c\tau+d)y,\tau)$$

とすると,$\Psi(y+1,\tau)=\Psi(y,\tau)$ となる. それは $\frac{\Psi(y+1,\tau)}{\Psi(y,\tau)}=1$ を示せばよく,

$$\theta((c\tau+d)(y+1),\tau)/\theta((c\tau+d)y,\tau)=\exp(\pi ic(c\tau+d)y^2)/\exp(\pi ic(c\tau+d)(y+1)^2)=\exp(-\pi ic(c\tau+d)(2y+1))$$

を示せば良い。また,

$$\begin{aligned}
\theta((c\tau+d)(y+1),\tau) &= \sum \exp(\pi in^2\tau+2\pi in(c\tau+d)y+2\pi in(c\tau+d)) && \exp(2\pi ind)=1\\
&= \sum \exp(\pi i(n+c)^2\tau-c^2\pi i\tau+2\pi in(c\tau+d)y) && \text{ただの展開}\\
&= \sum \exp(\pi i(n)^2\tau-c^2\pi i\tau+2\pi i(n-c)(c\tau+d)y) && y\text{ の置換}\\
&= \sum \exp(\pi i(n)^2\tau+2\pi in(c\tau+d)y-c^2\pi i\tau-2\pi iy(c\tau+d)(c))\\
&= \theta((c\tau+d)y,\tau)\exp(-c^2\pi i\tau-2\pi iy(c\tau+d)(c))\\
&= \theta((c\tau+d)y,\tau)\exp(-\pi ic(c\tau+d)-2\pi iy(c\tau+d)(c)) && \exp(-\pi icd)=1
\end{aligned}$$

より、確認できる.